Excitation by a transient signal of the real-valued electromagnetic fields in a cavity

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Excitation of electromagnetic fields in a cavity is studied in the time domain. A signal, which excites the fields, stands in Maxwell's equations as the electric current density given by an integrable function of coordinates and time. The problem is solved within the framework of the evolutionary approach to electromagnetics. The modal field expansions with time-dependent modal amplitudes are derived. Exact solutions for the amplitudes are obtained as the convolution integrals with time as a variable of integration, where the signal function stands as a parameter of the integrands. Two examples of the signal functions having a beginning in time are considered: (a) a surge modeled by the double-exponential function of time and (b) a sinusoid oscillating with an arbitrary frequency.

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I. INTRODUCTION

There are two goals in this article. One is to expose a version of the analytical approach to solving the system of Maxwell's equations with ∂_t, in which time is taken into consideration directly without resorting to the frequency Fourier analysis. In this version, the solution to the problem is obtained rigorously (in the mathematical sense) in the class of quadratically integrable real-valued functions of coordinates and time.

Time-domain studies of electromagnetics are of great importance nowadays. The greatest part of the results were obtained numerically in this area [1]. Finite-difference time-domain methods are able to provide a huge amount of numerical data. However, extraction of the physical content from those numerical data requires additional analytical information.

Historically, the greatest part of the analytical results were obtained by making use of Fourier and Laplace integral transforms. One can find rich information about the essential scientific achievements and a discussion of the advantages and disadvantages of such methods in [2]. An exhaustive historical background of this topic was presented as well in the fundamental work in [2].

Another way to approach analytical time-domain studies is connected with applications of ordinary and partial differential equations with time derivative in electromagnetic field theory [3–9]. One can find information about the historical background of this topic and a program for further development of this approach in Ref. [7]. Following that program, transient processes in cavities and waveguides were studied analytically; see an exposition of the first version of the approach and its implementations in the Refs. [4–8].

The second goal is to study transient electromagnetic fields, which can be excited in a cavity by an ultrawideband current surge. From a practical viewpoint, the results obtained may be useful, in particular, for the old problem of effective protection of microwave and electronic systems from voltage and current surges.

There are surges caused by multifarious natural phenomena like strokes of lightning, electrostatic discharges, breakdown effects in industrial and household appliances, switching, etc. They are able to encompass the dangerous failure of defense, navigation, and microwave systems, communication, computers, and others. Besides, similar surges can be caused by man-made sources. Nowadays, it has become possible to develop effective pulse and surge generators built in a very small volume. It has aroused anxiety with the prospect of their possible use as a tool for electromagnetic terrorism [10].

Surges are unavoidably accompanied by the onset of electromagnetic signals which are essentially distinct from habitual time-harmonic oscillations and waves. It requires adequate theoretical concepts for physical comprehension of these phenomena and practical operations afterwards. However, engineers have interpreted them mostly within the framework of the time-harmonic field concept which is dominant now. That is why even experts in this area estimate sometimes the actual state of affairs as "art rather than science." It seems topical to develop reliable fundamentals of electromagnetic field theory oriented in a study of such processes [11,12].

This paper is organized as follows. In Sec. II, a formulation of the problem is given, where the initial conditions for the fields and the causality principle are involved as well. In Sec. III, a general scheme of the method and its implementation are presented. In Sec. IV, exact solutions are obtained in the form of convolution integrals where a given signal function can be a variant by a researcher. In Sec. V, we draw our conclusions.

II. FORMULATION OF THE PROBLEM

Consider the system of Maxwell's equations

$$\nabla \times \mathcal{H}(\mathbf{r},t) = \varepsilon_0 \partial_t \mathcal{E}(\mathbf{r},t) + \sigma \mathcal{E}(\mathbf{r},t) + \mathcal{J}(\mathbf{r},t), \tag{1a}$$

$$\nabla \times \mathcal{E}(\mathbf{r},t) = -\mu_0 \partial_t \mathcal{H}(\mathbf{r},t), \tag{1b}$$

for the real-valued strengths of the electric and magnetic fields $\mathcal{E}$ and $\mathcal{H}$, respectively, where $\mathbf{r}$ is the position vector of

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a point of observation and t is an observation time. Equations (1) hold within a cavity volume V bounded by a closed singly connected (geometrically) perfectly conducting (physically) surface S. So the vectors $\mathcal{E}$ and $\mathcal{H}$ should satisfy the boundary conditions over S as

$$\mathbf{r} \in S, \quad \mathbf{n} \times \mathcal{E} = 0, \quad \mathbf{n} \cdot \mathcal{H} = 0, \tag{2}$$

where $\mathbf{n}$ is the unit vector outward normal to S. The term $\sigma\mathcal{E}$ (conductivity σ is a real-valued parameter) is involved in (1) for modeling the losses in the cavity.

The term $\mathcal{J}$ is added to Eq. (1a) for a description of a given signal which should excite an electromagnetic field in the cavity. Define it as follows:

$$\mathcal{J}(\mathbf{r},t) = [H(t)/T]\varepsilon_0 \mathcal{F}(\mathbf{r},t), \tag{3}$$

where $H(t)$ is the Heaviside step function. It means that the signal has a beginning in time at $t=0$. The parameter T is an intrinsic time which will be specified later on when needed. The factor $\mathcal{F}(\mathbf{r},t)$ is a given integrable real-valued vector function of space and time with the same physical dimension as the field $\mathcal{E}$. This term is assigned for a description of a signal carrier and a given wave form which excites the fields. It is appropriate to discuss the "signal function" $\mathcal{F}(\mathbf{r},t)$ in more detail later on, when it will be essential for implementation of the method of solution; see Sec. III E.

The formulation of time-domain problems should involve two additional mandatory statements, which are superfluous in the setting of time-harmonic field problems. (a) The initial conditions should be added like

$$\mathbf{r} \in V, \quad t=0: \quad \mathcal{E}(\mathbf{r},0) = 0, \quad \mathcal{H}(\mathbf{r},0) = 0, \tag{4}$$

inasmuch as Maxwell's equations with ∂_t require it as the partial differential equations of the hyperbolic kind. (b) The causality principle should be involved as

$$\mathcal{E}(\mathbf{r},t) = 0, \quad \mathcal{H}(\mathbf{r},t) = 0, \quad \text{if } t < 0, \tag{5}$$

as long as the "signal function" starts excitation of the cavity fields at the instant $t=0$, but is equal to zero while $t<0$; note that $\mathcal{J}(\mathbf{r},t)$ is proportional to $H(t)$.

The generally accepted physical statement that the energy of any electromagnetic field is always finite specifies a space of solutions to the problem (1)–(5) as follows:

$$\int_{t_1}^{t_2} dt \int_{V'} (\varepsilon_0 \mathcal{E} \cdot \mathbf{E} + \mu_0 \mathcal{H} \cdot \mathbf{H}) dv < \infty, \tag{6}$$

where $0 \leq t_1 < t_2 < \infty$, $V' \subseteq V$, and the centerdot stands for scalar multiplication of the vectors. Hence, the solution to the problem should be found in Hilbert space L_2.

III. EVOLUTIONARY APPROACH TO THE STUDIES OF *REAL-VALUED* ELECTROMAGNETIC FIELDS

A. General scheme of the approach

Let us imagine that Eqs. (1) and (2) appear as the result of an action on the vectors $\mathcal{E}$ and $\mathcal{H}$ of an operator $\mathcal{M}$ provided that it is composed of two parts as $\mathcal{M} = \mathfrak{R} + \mathcal{A}$. Suppose that the operator $\mathfrak{R}$ has the following properties: (a) it is *linear* under construction, (b) it acts on the coordinates solely, and (c) it is self-adjoint in the space of solutions L_2. The operator $\mathcal{A}$ is the remainder of $\mathcal{M}$, and the time derivative ∂_t refers to the operator $\mathcal{A}$.

Due to the self-adjointness of the operator $\mathfrak{R}$, the set of its eigenvectors is complete and it originates as a basis in L_2. The elements of the basis have a physical sense of the cavity modes: solenoidal and irrotational both.

The operator $\mathfrak{R}$ is diagonal because it is self-adjoint. It can be inverted analytically, which yields the eigenvector series. The series has the physical sense of modal expansions for the vectors $\mathcal{E}$ and $\mathcal{H}$. The coefficients in the series are the time-dependent modal amplitudes.

The operator $\mathcal{A}$ (where ∂_t was saved) supplies a problem for the modal amplitudes. It is the well-studied, in mathematics, Cauchy problem for a system of differential equations with time derivative. Mathematicians call all differential equations with time derivative *evolutionary equations*. Indeed, any solution to the Cauchy problem shows how the process progresses in time (i.e., evolves) from its state given at the initial instant to the state at an observation time. That is why we name this method the *evolutionary approach to electromagnetics*.

The approach, which can be implemented by this scheme, is prospective for the development of electromagnetic field theory just in the time domain directly. It enables one to involve in electromagnetics the vast theory of evolutionary equations as a powerful tool aimed at studies of the various transient processes.

B. Self-adjoint operator

Introduce a real-valued six-component vector $\mathcal{X}$ composed of the fields $\mathcal{E}$ and $\mathcal{H}$ as follows:

$$\mathcal{X}(\mathbf{r},t) = \mathrm{col}(\mathcal{E}(\mathbf{r},t), \mathcal{H}(\mathbf{r},t)), \tag{7}$$

where "col" means column here and henceforward. Consider the *pair* of three-component vectors, Eqs. (1), as single six-component vector equation. Then Eqs. (1) can be briefly written as $\mathfrak{R}'\mathcal{X}(\mathbf{r},t) = \mathcal{A}\mathcal{X}(\mathbf{r},t)$. The notation $\mathfrak{R}'$ signifies a 6×6 matrix differential procedure:

$$\mathfrak{R}'\mathcal{X}(\mathbf{r},t) = \begin{pmatrix} \mathcal{O} & \varepsilon_0^{-1}\nabla\times \\ \mu_0^{-1}\nabla\times & \mathcal{O} \end{pmatrix} \begin{pmatrix} \mathcal{E}(\mathbf{r},t) \\ \mathcal{H}(\mathbf{r},t) \end{pmatrix}, \tag{8}$$

where $\mathbf{r} \notin S$. The notation $\mathcal{O}$ means a 3×3 zero-valued matrix. Then the right-hand sides of Eqs. (1) can be denoted as the six-component vector $\mathcal{A}\mathcal{X}(\mathbf{r},t)$ introduced as follows:

$$\mathcal{A}\mathcal{X}(\mathbf{r},t) = \begin{pmatrix} \partial_t\mathcal{E} + (\sigma/\varepsilon_0)\mathcal{E} + \varepsilon_0^{-1}\mathcal{J} \\ -\partial_t\mathcal{H} \end{pmatrix}. \tag{9}$$

Aggregation of the differential procedure $\mathfrak{R}'$ and the boundary conditions (2) yields a bounded operator as

$$\mathfrak{R}\mathcal{X}(\mathbf{r},t) = \begin{cases} \mathfrak{R}'\mathcal{X}(\mathbf{r},t), & \mathbf{r} \in V, \quad \mathbf{r} \notin S, \\ \mathbf{n} \times \mathcal{E} = 0, \quad \mathbf{n} \cdot \mathcal{H} = 0, & \mathbf{r} \in S. \end{cases} \tag{10}$$

Previously, a similar operator was introduced with involvement of the imaginary unit i in the definition of the differential procedure $\mathfrak{R}'$; see [3,5,7,8]. Therefore, we cannot use it in these studies of real-valued fields.

It is evident that time t plays the role of a parameter in regard to the action of the operator $\mathfrak{R}$. It suggests to put into play a space of six-component real vectors with elements

$$\mathfrak{X}(\mathbf{r}) = \text{col}(\mathbf{E}(\mathbf{r}), \mathbf{H}(\mathbf{r})), \tag{11}$$

the constituents of which depend on coordinates $(\mathbf{r})$ solely and satisfy the same boundary conditions as in (10):

$$\mathbf{n} \times \mathbf{E}(\mathbf{r}) = 0, \quad \mathbf{n} \cdot \mathbf{H}(\mathbf{r}) = 0, \quad \mathbf{r} \in S. \tag{12}$$

Specify this space by introducing the inner product as

$$\langle \mathfrak{X}_1, \mathfrak{X}_2 \rangle = \frac{1}{V} \int_V (\varepsilon_0 \mathbf{E}_1 \cdot \mathbf{E}_2 + \mu_0 \mathbf{H}_1 \cdot \mathbf{H}_2) dv, \tag{13}$$

where the pair $\mathfrak{X}_1 = \text{col}(\mathbf{E}_1, \mathbf{H}_1)$, $\mathfrak{X}_2 = \text{col}(\mathbf{E}_2, \mathbf{H}_2)$ is composed of any elements from the space.

One can make certain that the identity holds as

$$\langle \mathfrak{R}\mathfrak{X}_1, \mathfrak{X}_2 \rangle - \langle \mathfrak{X}_1, \mathfrak{R}\mathfrak{X}_2 \rangle = 0. \tag{14}$$

Hence, the operator $\mathfrak{R}$ is self-adjoint in L_2 and the following operator eigenvalue equation holds as

$$\mathfrak{R}\mathfrak{X}_n(\mathbf{r}) = \omega_n \mathfrak{X}_n(\mathbf{r}), \tag{15}$$

where the ω_n's are eigenvalues of the operator $\mathfrak{R}$ and the $\mathfrak{X}_n$'s are its eigenvectors corresponding to these eigenvalues. The subscript n $(n=0, \pm 1, \pm 2, \ldots)$ regulates the positions of the eigenvalues on a real axis $O\omega$ in order of increasing numerical values.

With making use of that identity (14), one can prove that (a) all the eigenvalues ω_n's are real valued, (b) the spectrum $\{\omega_n\}_{n=-\infty}^{\infty}$ is discrete because the domain V is finite, and (c) any pair of the eigenvectors $\mathfrak{X}_n$ and $\mathfrak{X}_{n'}$, which correspond to distinct eigenvalues ω_n and $\omega_{n'}$, respectively, are orthogonal in the sense $\langle \mathfrak{X}_n, \mathfrak{X}_{n'} \rangle = 0$.

It is appropriate to emphasize here that Eqs. (14) and (15) have been derived by making use of the differential procedure $\mathfrak{R}'$ and the boundary conditions (12) only. The time derivative ∂_t was referred to the vector (9) and retained there while these derivations were performed.

C. Modal basis in the space of solutions

Substitution of the operator $\mathfrak{R}$ to Eq. (15) yields

$$\nabla \times \mathbf{H}_n = \omega_n \varepsilon_0 \mathbf{E}_n, \quad (\mathbf{n} \cdot \mathbf{H}_n)|_S = 0, \tag{16a}$$

$$\nabla \times \mathbf{E}_n = \omega_n \mu_0 \mathbf{H}_n, \quad [\mathbf{n} \times \mathbf{E}_n]|_S = 0. \tag{16b}$$

By making use of Stocks' theorem and provided that $\omega_n \neq 0$, the equation $\nabla \times \mathbf{E}_n = \omega_n \mu_0 \mathbf{H}_n$ shows that the condition $(\mathbf{n} \cdot \mathbf{H}_n)|_S = 0$ holds automatically if the condition $[\mathbf{n} \times \mathbf{E}_n]|_S = 0$ holds and vice versa. Hence, the problem (16) contains ultimately two different boundary eigenvalue problems. One of them is

$$\nabla \times \mathbf{H}_n'(\mathbf{r}) = \omega_n' \varepsilon_0 \mathbf{E}_n'(\mathbf{r}),$$

$$\nabla \times \mathbf{E}_n'(\mathbf{r}) = \omega_n' \mu_0 \mathbf{H}_n'(\mathbf{r}),$$

$$[\mathbf{n} \times \mathbf{E}_n']|_S = 0, \tag{17}$$

where $\omega_n' \neq 0$ are real eigenvalues. The other one is

$$\nabla \times \mathbf{H}_n''(\mathbf{r}) = \omega_n'' \varepsilon_0 \mathbf{E}_n''(\mathbf{r}),$$

$$\nabla \times \mathbf{E}_n''(\mathbf{r}) = \omega_n'' \mu_0 \mathbf{H}_n''(\mathbf{r}),$$

$$(\mathbf{n} \cdot \mathbf{H}_n'')|_S = 0, \tag{18}$$

where $\omega_n'' \neq 0$ is another real eigenvalue. In the general case, the sets $\{\omega_n'\}$ and $\{\omega_n''\}$ are different. Solving the problems (17) and (18) yields the real-valued eigenfunctions.

If the cavity volume V is performed as a shorted piece of a cylinder with a rather arbitrary cross section, all the solutions to the problem (17) can be associated with a complete set of the solenoidal TE modes. Then all the solutions to the problem (18) can be regarded to another complete set of the solenoidal TM modes.

In our previous versions of the method, the problem (16) was obtained with the imaginary unit i as a factor on the right-hand side of the second differential equation and with the factor $-i$ on the right-hand side of the first equation. Surely, it cannot support the real-valued solutions.

Normalize all the solutions for these problems as

$$\frac{\varepsilon_0}{V} \int_V \mathbf{E}_n \cdot \mathbf{E}_n dv = \frac{\mu_0}{V} \int_V \mathbf{H}_n \cdot \mathbf{H}_n dv = 1. \tag{19}$$

Substitute now $n=0$ and $\omega_0=0$ into the problem (16a) and (16b). It yields a pair of uncoupled boundary value problems as

$$\nabla \times \mathbf{E}_0 = \mathbf{0}, \quad [\mathbf{n} \times \mathbf{E}_0]|_S = 0, \tag{20}$$

where $\mathbf{0}$ is three-component zero-valued vector and

$$\nabla \times \mathbf{H}_0 = \mathbf{0}, \quad (\mathbf{n} \cdot \mathbf{H}_0)|_S = 0. \tag{21}$$

The problems (20) and (21) suggest that some eigenvectors like $\text{col}(\mathbf{E}_0(\mathbf{r}), \mathbf{0})$ and $\text{col}(\mathbf{0}, H_0(\mathbf{r}))$ can correspond to the eigenvalue $\omega_0=0$ of the operator $\mathfrak{R}$.

The differential equations (20) and (21) in their old notation as rot $\mathbf{E}_0=\mathbf{0}$ and rot $\mathbf{H}_0=\mathbf{0}$ gave the name to the vectors $\mathbf{E}_0$ and $\mathbf{H}_0$ as *irrotational* ones.

The vectors $\mathbf{E}_0(\mathbf{r})$ and $\mathbf{H}_0(\mathbf{r})$ are presentable via the scalar potentials ϕ and ψ as $\mathbf{E}_0 = \nabla\phi$ and $\mathbf{H}_0 = \nabla\psi$ where $\phi(\mathbf{r})$ and $\psi(\mathbf{r})$ can be arbitrary twice differentiable functions varying within the domain V. The arbitrary rule in the choice of the potentials $\phi(\mathbf{r})$ and $\psi(\mathbf{r})$ suggests that the eigenvalue $\omega_0=0$ has infinite multiplicity.

The conditions $[\mathbf{n} \times \mathbf{E}_0]|_S = 0$ and $(\mathbf{n} \cdot \mathbf{H}_0)|_S = 0$ yield the boundary conditions for the potentials as $\phi|_S = 0$ and $\partial_{\mathbf{n}} \psi|_S = 0$, respectively, where $\partial_{\mathbf{n}}$ is the normal to the S derivative. Remembering the Sturm-Liouville theorems from mathematical physics, just the boundary conditions $\phi|_S = 0$ and $\partial_{\mathbf{n}} \psi|_S = 0$ suggest one to supplement each one with Helmholtz equations. In such a way, one can obtain two different complete sets of normalized potentials $\{\phi_n(\mathbf{r})\}_{n=1}^{\infty}$ and $\{\psi_n(\mathbf{r})\}_{n=0}^{\infty}$ as solutions to the well-studied Dirichlet and Neumann boundary eigenvalue problems for Laplacians, respectively, which are set as

$$\nabla^2\phi_n(\mathbf{r}) + \kappa_n^2\phi_n(\mathbf{r}) = 0,$$

$$\phi_n|_S = 0,$$

$$\frac{\varepsilon_0\kappa_n^2}{V}\int_V |\phi_n|^2 dv = 1, \quad (22)$$

where $\kappa_n^2 > 0$ are the eigenvalues, $n=1,2,\ldots$, and the ψ_n's are the real-valued solutions to Neumann problem (22):

$$\nabla^2\psi_n(\mathbf{r}) + \nu_n^2\psi_n(\mathbf{r}) = 0,$$

$$\partial_{\mathbf{n}}\,\psi_n|_S = 0,$$

$$\frac{\varepsilon_0^2\nu_n^2}{V}\int_V |\psi_n|^2 dv = 1, \quad (23)$$

where $\nu_n^2 \geq 0$ are the eigenvalues, $n=0,1,2,\ldots$. The solution $\psi_0(\mathbf{r})$ is a harmonic function since it corresponds to the eigenvalue $\nu_0^2=0$. It is evident that $\psi_0(\mathbf{r})$ (where $\mathbf{r}\in S,V$) is a constant and, hence, $\nabla\psi_0=0$. It is true if the cavity surface S is a singly connected domain only.

The normalization conditions for the potentials, which were introduced in Eqs. (22) and (23), generate the normalizations for the irrotational vectors $\nabla\phi_n$ and $\nabla\psi_n$ as

$$\frac{\varepsilon_0}{V}\int_V |\nabla\phi_n|^2 dv = \frac{\mu_0}{V}\int_V |\nabla\psi_n|^2 dv = 1, \quad (24)$$

which coincides with the normalization conditions (19) adopted above for the solenoidal vectors.

Thus, a set $\mathfrak{L}=\{\mathfrak{X}_n(\mathbf{r})\}_{n=-\infty}^{+\infty}$ of eigenvectors $\mathfrak{X}_n$ corresponding to the set of eigenvalues $\{\omega_n\}_{n=-\infty}^{+\infty}$ of the operator $\mathfrak{R}$, has been established. The boundary eigenvalue problems (17), (18), (22), and (23) can be interpreted as a general form of the definitions for the elements of $\mathfrak{L}$. Let us prove now that the set $\mathfrak{L}$ is complete and it originates a basis in the space of solutions.

The set $\mathfrak{L}$ [with the eigenvectors $\mathfrak{X}_n=\mathrm{col}(\mathbf{E}_n,\mathbf{H}_n)$ as its elements] can be presented in vector form as $\mathfrak{L}=\mathrm{col}(\mathfrak{E},\mathfrak{H})$. The subspaces $\mathfrak{E}$ and $\mathfrak{H}$ are composed of the three-component constituents of the vectors $\mathfrak{X}_n$'s as

$$\mathfrak{E} = \{\mathbf{E}_n'(\mathbf{r})\} \oplus \{\mathbf{E}_n''(\mathbf{r})\} \oplus \{\nabla\phi_n(\mathbf{r})\},$$

$$\mathfrak{H} = \{\mathbf{H}_n'(\mathbf{r})\} \oplus \{\mathbf{H}_n''(\mathbf{r})\} \oplus \{\nabla\psi_n(\mathbf{r})\}, \quad (25)$$

where $\oplus$ stands for direct summation of the subsubspaces in the subspaces $\mathfrak{E}$ and $\mathfrak{H}$, which are composed of the solenoidal and irrotational vectors as their elements.

The Weyl theorem (about orthogonal detachments of Hilbert space L_2) says that the subspaces $\mathfrak{E}$ and $\mathfrak{H}$ are both complete in L_2 [13]. Hence the set $\mathfrak{L}$, which is composed of the subspaces $\mathfrak{E}$ and $\mathfrak{H}$, is complete in L_2 as well and it originates a basis. The elements of $\mathfrak{E}$ and $\mathfrak{H}$ [see (25)] have the physical sense of the cavity modes. So it is natural to call that basis a *modal* basis in accordance with the physical sense of its elements.

D. Modal basis for a cylindrical cavity

Let us consider a case when the cavity volume V is a shorted piece of a waveguide which is geometrically homogeneous along its axis—say, Oz. The waveguide cross section $S_\perp$ may be bounded by an arbitrary (but enough smooth, however) closed contour L. The term "enough smooth contour" implies that none of its possible inner angles (i.e., measured within $S_\perp$) exceeds π.

Present the position vector $\mathbf{r}$ and nabla operator ∇ as

$$\mathbf{r} = \mathbf{r}_\perp + \mathbf{z}z, \qquad \nabla = \nabla_\perp + \mathbf{z}\partial_z, \quad (26)$$

where $\mathbf{r}_\perp$ and $\nabla_\perp$ are the projections on the domain $S_\perp$ and $\mathbf{z}$ is the unit vector oriented along the axis Oz. Define the cavity volume V: $\mathbf{r}_\perp\in S_\perp$, $-\ell \leq z \leq +\ell$, where 2ℓ is the distance between the face planes of the cavity.

In a similar way, present the three-component elements $\mathbf{E}_n(\mathbf{r})$ and $\mathbf{H}_n(\mathbf{r})$ of the modal basis as follows:

$$\mathbf{E}_n(\mathbf{r}) = \mathbf{E}_{\perp n}(\mathbf{r}_\perp,z) + \mathbf{z}E_{zn}(\mathbf{r}_\perp,z),$$

$$\mathbf{H}_n(\mathbf{r}) = \mathbf{H}_{\perp n}(\mathbf{r}_\perp,z) + \mathbf{z}H_{zn}(\mathbf{r}_\perp,z). \quad (27)$$

Below, the boundary eigenvalue problems (17), (18), (22), and (23) will be solved successively by separation of the pair of transverse coordinates ($\mathbf{r}_\perp$) and variable z.

1. Subspace of TE modes in the modal basis

Solving the problem (17) provided that $E_{zn}\equiv 0$ yields

$$\mathbf{E}_{\perp n}'(\mathbf{r}_\perp,z) = A_n'\omega_n'\mu_0[\nabla_\perp\vartheta_{pq}(\mathbf{r}_\perp)\times\mathbf{z}]h_l(z),$$

$$\mathbf{H}_{\perp n}'(\mathbf{r}_\perp,z) = A_n'\nabla_\perp\vartheta_{pq}(\mathbf{r}_\perp)\partial_z h_l(z),$$

$$H_{zn}'(\mathbf{r}_\perp,z) = A_n'\nu_{\perp pq}^2\vartheta_{pq}(\mathbf{r}_\perp)h_l(z), \quad (28)$$

where A_n' is a normalization constant. The set of functions $\{\vartheta_{pq}(\mathbf{r}_\perp)\}$ is complete since it is composed of all the solutions to Neumann boundary eigenvalue problem for transverse Laplacian $\nabla_\perp^2$ which is set as follows:

$$\nabla_\perp^2\vartheta_{pq}(\mathbf{r}_\perp) + \nu_{\perp pq}^2\vartheta_{pq}(\mathbf{r}_\perp) = 0,$$

$$\partial_{\mathbf{n}}\vartheta_{pq}(\mathbf{r}_\perp)|_L = 0, \quad (29)$$

where $\nu_{\perp pq}^2 > 0$ are the eigenvalues and $\partial_{\mathbf{n}}$ is the normal to the domain $S_\perp$ derivative on the contour L. The solutions to the problem (29) are often called "membrane functions." The factor $h_l(z)$ can be found by solving the following simple boundary eigenvalue problem:

$$\partial_z^2 h_l(z) + k_l^2 h_l(z) = 0,$$

$$h_l(z)|_{z=\pm\ell} = 0. \quad (30)$$

where $k_l^2 > 0$ are the eigenvalues. The problem (30) yields two sets of solutions: even (e) and odd (o) with respect to the axis Oz. The first set involves all the functions $h_l^e(z) = \cos[\pi(l+\frac{1}{2})z/\ell]$. They correspond to the eigenvalues $k_l^2 = [\pi(l+\frac{1}{2})/\ell]^2$, $l=0,1,2,\ldots$. The second set is composed of the functions $h_l^o(z)=\sin(\pi l z/\ell)$, which correspond to the eigenvalues $k_l^2=(\pi l/\ell)^2$, $l=1,2,\ldots$.

The eigenvalues ω_n''s of the operator $\mathfrak{R}$ are equal to

$$\omega_n' \equiv \omega_{pql}' = c\sqrt{\nu_{\perp pq}^2 + k_l^2}, \tag{31}$$

where $c=1/\sqrt{\varepsilon_0\mu_0}$. The real numbers $\nu_{\perp pq}^2$ should be found in the process of solving the problem (29) in which the contour L should be specified.

Remark 1. We introduced above the subscript n as a marker which regulates the position of the eigenvalues ω_n's of the operator $\mathfrak{R}$ on a real axis in order of increasing numerical values. In solving the practical examples, the subscript n exhibits itself as a triplet: $n\to(p,q,l)$ where p, q, and l are independent parameters.

Here are two elementary examples on this topic.

Example 1. The contour L is rectangular: $0\le x\le a$, $0\le y\le b$. This contour is smooth enough for our theory. Solving the problem (29) yields $\nu_{\perp pq}^2=(\pi p/a)^2+(\pi q/b)^2$, $\vartheta_{pq}(\mathbf{r}_\perp)\equiv\vartheta_{pq}(x,y)=\cos(\pi px/a)\cos(\pi qy/b)$, where p and q are integers, $p,q=0,1,2,\ldots$, provided that $p+q\neq0$.

Example 2. The contour L is circular: $0\le\alpha\le2\pi$, $0\le r\le a$, where a is the radius of the circle. The numbers $\nu_{\perp pq}$ are the pth roots of the derivative of a Bessel function of order $q=0,1,2,\ldots$, $\frac{d}{dr}J_q\,(\nu_{\perp pq}\frac{r}{a})|_{r=a}=0$, provided that

$$\vartheta_{pq}(r,\alpha) = J_q\left(\nu_{\perp pq}\frac{r}{a}\right)\begin{matrix}\swarrow & \sin(q\alpha), \quad q\neq0.\\ \nwarrow & \cos(q\alpha),\end{matrix} \tag{32}$$

The constants A_n' can be calculated by formula (19).

2. Subspace of TM modes in the modal basis

Solving the problem (18) provided that $H_{zn}\equiv0$ yields

$$\mathbf{H}_{\perp n}''(\mathbf{r}_\perp,z) = A_n''\omega_n''\varepsilon_0[\boldsymbol{\nabla}_\perp\varphi_{pq}(\mathbf{r}_\perp)\times\mathbf{z}]e_l(z),$$

$$\mathbf{E}_{\perp n}''(\mathbf{r}_\perp,z) = A_n''\boldsymbol{\nabla}_\perp\varphi_{pq}(\mathbf{r}_\perp)\partial_z e_l(z),$$

$$E_{zn}''(\mathbf{r}_\perp,z) = A_n''\kappa_{\perp pq}^2\varphi_{pq}(\mathbf{r}_\perp)e_l(z), \tag{33}$$

where A_n'' is a normalization constant. In this case, another complete set of membrane functions $\{\varphi_{pq}(\mathbf{r}_\perp)\}$ should be found by solving the Dirichlet boundary eigenvalue problem for the transverse Laplacian $\nabla_\perp^2$ as

$$\nabla_\perp^2\varphi_{pq}(\mathbf{r}_\perp) + \kappa_{\perp pq}^2\varphi_{pq}(\mathbf{r}_\perp) = 0,$$

$$\varphi_{pq}\,(\mathbf{r}_\perp)|_L = 0, \tag{34}$$

where $\kappa_{\perp pq}^2>0$ are the eigenvalues. In turn, the set of functions $\{e_l(z)\}$ is composed of all solutions to the following boundary eigenvalue problem:

$$\partial_z^2 e_l(z) + \hat{k}_l^2 e_l(z) = 0,$$

$$\partial_z e_l\,(z)|_{z=\pm\ell} = 0, \tag{35}$$

where $\hat{k}_l^2>0$ are the eigenvalues. This problem yields two infinite sets of solutions: $e_l^o(z)=\sin[\pi(l+\frac{1}{2})z/\ell]$, which correspond to the eigenvalues $\hat{k}_l^2=[\pi(l+\frac{1}{2})/\ell]^2$ with $l=0,1,2,\ldots$, and $e_l^e(z)=\cos(\pi lz/\ell)$, which correspond to the eigenvalues $\hat{k}_l^2=(\pi l/\ell)^2$ with $l=0,1,2,\ldots$.

The eigenvalues ω_n'' of the operator $\mathfrak{R}$ are equal to

$$\omega_n'' \equiv \omega_{pql}'' = c\sqrt{\kappa_{\perp pq}^2 + \hat{k}_l^2}. \tag{36}$$

Continue our examples for the chosen forms of L.

Example 3. In the case of the rectangular contour L, the problem (34) yields the set of functions $\varphi_{pq}(\mathbf{r}_\perp)=\sin(\pi px/a)\sin(\pi qy/b)$ corresponding to the eigenvalues $\kappa_{\perp pq}^2=(\pi p/a)^2+(\pi q/b)^2$ with $p,q=1,2,\ldots$.

Example 4. If the contour L is that circle, the numbers $\kappa_{\perp pq}>0$ are the pth roots $(p=1,2,\ldots)$ of the Bessel function of order q: $J_q(\kappa_{\perp pq})=0$, $q=1,2,\ldots$. The membrane functions $\varphi_{pq}(\mathbf{r}_\perp)$ are specified as follows:

$$\varphi_{pq}(r,\alpha) = J_q\left(\kappa_{\perp pq}\frac{r}{a}\right)\begin{matrix}\swarrow & \sin(q\alpha), \quad q\neq0.\\ \nwarrow & \cos(q\alpha),\end{matrix} \tag{37}$$

The constants A_n'' can be calculated by formula (19).

3. Subspace of irrotational modes of the electric kind

Denote the transverse and axial parts of the irrotational vectors $\boldsymbol{\nabla}\phi_n$ in accordance with the notation (27), respectively, as follows: $\mathbf{E}_{\perp n}=\boldsymbol{\nabla}_\perp\phi_n$ and $E_{zn}=\partial_z\phi_n$. Solving the problem (22) for them results in

$$\mathbf{E}_{\perp n}(\mathbf{r}_\perp,z) = A_n\boldsymbol{\nabla}_\perp\varphi_{pq}(\mathbf{r}_\perp)h_l(z),$$

$$E_{zn}(\mathbf{r}_\perp,z) = A_n\varphi_{pq}(\mathbf{r}_\perp)\partial_z h_l(z), \tag{38}$$

where A_n is a normalization constant. The membrane functions $\varphi_{pq}(\mathbf{r}_\perp)$ here are the same as in the problem (34). The factors $h_l(z)$ here are the same which the problem (30) yields. One can calculate the constants A_n by the normalization condition exhibited in (22). The numbers κ_n^2 needed for this aim are $\kappa_n^2=\sqrt{\kappa_{\perp pq}^2+k_l^2}$.

4. Subspace of the magnetic irrotational modes

Introduce the analogous notation $\boldsymbol{\nabla}\psi_n=H_{\perp n}+zH_{zn}$ and solve the problem (23). It yields

$$\mathbf{H}_{\perp n}(\mathbf{r}_\perp,z) = B_n\boldsymbol{\nabla}_\perp\vartheta_{pq}(\mathbf{r}_\perp)e_l(z),$$

$$H_{zn}(\mathbf{r}_\perp,z) = B_n\vartheta_{pq}(\mathbf{r}_\perp)\partial_z e_l(z), \tag{39}$$

where B_n is a normalization constant. The membrane functions $\vartheta_{pq}(\mathbf{r}_\perp)$ here are the same as in the problem (29). The factors $e_l(z)$ here are the same which the problem (35) yields. One can calculate the constants B_n by the normalization condition exhibited in (23). The numbers ν_n^2 needed for this aim are $\nu_n^2=\sqrt{\nu_{\perp pq}^2+\hat{k}_l^2}$.

E. Modal expansions for the real-valued fields

Projecting the fields $\mathcal{E}$ and $\mathcal{H}$ onto the elements of the modal basis yields the modal expansions as

$$\mathcal{E}(\mathbf{r},t) = \sum_{n=1}^{\infty} e_n'\mathbf{E}_n' + \sum_{n=1}^{\infty} e_n''\mathbf{E}_n'' + \sum_{n=1}^{\infty} a_n\,\boldsymbol{\nabla}\,\phi_n,$$

$$\mathcal{H}(\mathbf{r},t) = \sum_{n=1}^{\infty} h_n \mathbf{H}_n + \sum_{n=1}^{\infty} h_n'' \mathbf{H}_n'' + \sum_{n=1}^{\infty} b_n \nabla \psi_n, \quad (40)$$

where the vectorial basis elements depend on coordinates solely. Hence, the scalar coefficients e_n, e_n'', a_n and h_n, h_n'', b_n must be time dependent because the left-hand sides of Eqs. (40) depend on time. The scalar coefficients can be named the *modal amplitudes*, physically.

It is appropriate to indicate here that the series (40) satisfy the boundary conditions (2) because every element of the basis satisfies the same conditions (12).

The given force function $\mathcal{F}(\mathbf{r},t)$, which was incorporated into $\mathcal{J}(\mathbf{r},t)$ [see (3)], can be projected onto the basis elements just in the same way as the field $\mathcal{E}$. It yields

$$\mathcal{F} = \sum_{n=1}^{\infty} \mathcal{F}_n'(t)\mathbf{E}_n' + \sum_{n=1}^{\infty} \mathcal{F}_n''(t)\mathbf{E}_n'' + \sum_{n=1}^{\infty} \mathcal{F}_n^0(t) \nabla \phi_n, \quad (41)$$

where the modal amplitudes are known as

$$\mathcal{F}_n'(t) = \frac{\varepsilon_0}{V}\int_V \mathcal{F}(\mathbf{r},t)\cdot \mathbf{E}_n'(\mathbf{r})dv,$$

$$\mathcal{F}_n''(t) = \frac{\varepsilon_0}{V}\int_V \mathcal{F}(\mathbf{r},t)\cdot \mathbf{E}_n''(\mathbf{r})dv,$$

$$\mathcal{F}_n^0(t) = \frac{\varepsilon_0}{V}\int_V \mathcal{F}(\mathbf{r},t)\cdot \nabla\phi_n(\mathbf{r})dv. \quad (42)$$

If the force function $\mathcal{F}(\mathbf{r},t)$ corresponds to a spot dipole, which supplies a given signal $\mathfrak{s}(t)$ to the cavity, then

$$\mathcal{F}(\mathbf{r},t) = \mathbf{d}\,\delta(\mathbf{r}-\mathbf{r}_s)\mathfrak{s}(t), \quad (43)$$

where $\delta(\mathbf{r}-\mathbf{r}_s)$ is the Dirac delta-function, the position vector $\mathbf{r}_s$ specifies a location of the dipole within the cavity, and the vector $\mathbf{d}$ specifies its orientation and intensity measured in the same unit which the field $\mathcal{E}$ has. The dimensionless signal $\mathfrak{s}(t)$ may be an arbitrary integrable function of time. Substitution of formula (43) into Eqs. (42) yields

$$\mathcal{F}_n'(t) = F_n'\mathfrak{s}(t), \quad \mathcal{F}_n''(t) = F_n''\mathfrak{s}(t), \quad \mathcal{F}_n^0(t) = F_n^0\mathfrak{s}(t),$$

$$F_n' = \frac{\varepsilon_0}{V}\mathbf{d}\cdot\mathbf{E}_n'(\mathbf{r}_s), \quad F_n'' = \frac{\varepsilon_0}{V}\mathbf{d}\cdot\mathbf{E}_n''(\mathbf{r}_s), \quad F_n^0 = \frac{\varepsilon_0}{V}\mathbf{d}\cdot\nabla\phi_n(\mathbf{r}_s)$$

where F_n', F_n'', and F_n^0 are constants.

F. Modal evolutionary equations

The identity (14) holds for any pair of vectors from the space of solutions. Take the vector $\mathcal{X}=\mathrm{col}(\mathcal{E},\mathcal{H})$ as the vector $\mathfrak{X}_1$. The field expansions (40) can be interpreted as the definitions of the vector $\mathcal{X}$ components. Instead of the vector $\mathfrak{R}\mathfrak{X}_1=\mathfrak{R}\mathcal{X}$, take the right-hand side of Eq. (9) (where ∂_t has been retained so far). As the vector $\mathfrak{X}_2$, take sequentially all the eigenvectors of the operator $\mathfrak{R}$. Use the operator (15) as a direct formula for the vector $\mathfrak{R}\mathfrak{X}_2$. These simple algebraic manipulations result finally in a set of evolutionary ordinary differential equations for the modal amplitudes.

Projecting the initial conditions (4) for the fields $\mathcal{E}$ and $\mathcal{H}$ onto the modal basis yields the initial conditions for the modal amplitudes. Combination of the evolutionary equations with appropriate initial conditions yields a set of uncoupled Cauchy problems for the modal amplitudes:

$$\frac{d}{dt}e_n' + 2\gamma e_n' - \omega_n' h_n' = -F_n' H(t)\frac{\mathfrak{s}(t)}{T}, \quad e_n'(0)=0, \quad (44a)$$

$$\frac{d}{dt}h_n' + \omega_n' e_n' = 0, \quad h_n'(0)=0, \quad (44b)$$

$$\frac{d}{dt}e_n'' + 2\gamma e_n'' - \omega_n'' h_n'' = -F_n'' H(t)\frac{\mathfrak{s}(t)}{T}, \quad e_n''(0)=0, \quad (45a)$$

$$\frac{d}{dt}h_n'' + \omega_n'' e_n'' = 0, \quad h_n''(0)=0, \quad (45b)$$

$$\frac{d}{dt}a_n + 2\gamma a_n = -F_n^0 H(t)\frac{\mathfrak{s}(t)}{T}, \quad a_n(0)=0, \quad (46)$$

$$\frac{d}{dt}b_n = 0, \quad b_n(0)=0, \quad (47)$$

where $\gamma=\sigma/(2\varepsilon_0)$ is the lossy parameter. For the sake of simplicity, we take the case of the spot dipole (43) when $\mathcal{F}_n'(t)=F_n'\mathfrak{s}(t)$, $\mathcal{F}_n''(t)=F_n''\mathfrak{s}(t)$, and $\mathcal{F}_n^0(t)=F_n^0\mathfrak{s}(t)$.

It is evident that every problem from the set (44)–(47) has a real-valued solution. In our previous studies, similar differential equations were derived with participation of the operator $\mathfrak{R}$, which enclosed imaginary unit i. Surely, the modal amplitudes were obtained as the complex-valued quantities; see [3,5,7,8].

Observation of the problems (44)–(47) enables us to express several conclusions without solving them.

(i) Our derivation of the evolutionary equations is irrelevant to the manner of definition of the signal $\mathfrak{s}(t)$. Hence, one can take even values of $\mathfrak{s}(t)$ detected experimentally.

(ii) Physically, the subscript n can be interpreted as an identifier for the cavity modes. Every problem from the set (44)–(47) involves only one value of n parametrically. Besides, these problems are uncoupled. Hence, every mode evolves individually starting from its initial value. It means, in turn, that the modal expansions (40) satisfy the initial conditions (4) automatically.

(iii) If $t<0$, all right-hand sides of the differential equations (44)–(46) become zero due to $H(t)$. All initial conditions are zeros. So every modal amplitude is equal to zero while $t<0$. Hence, the modal expansions (40) satisfy the causality principle (5) automatically.

(iv) The right-hand side in problem (46) is proportional to $\mathfrak{s}(t)$. Hence, solutions for the amplitudes $a_n(t)$ of the irrotational modes should be time dependent. However, the classical time-harmonic field concept always interprets all the irrotational modes as a static field.

(v) Problem (47) has an evident solution: $b_n(t)=0$. It is so because an extrinsic force of the magnetic kind is absent now

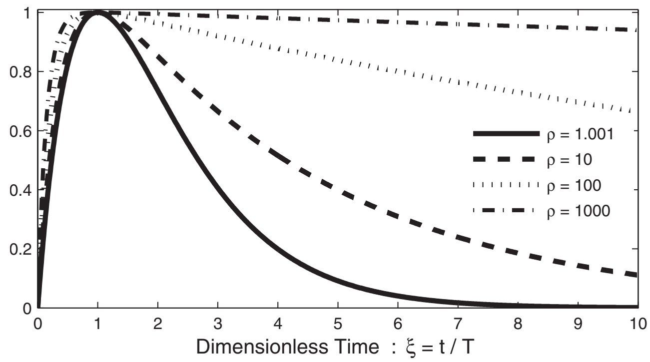

FIG. 1. Dependence on time of the surge for several values of ρ.

in Eq. (1b). However, if such a force is present—say, $\mathcal{I}(\mathbf{r},t)$—then a time-dependent projection on the appropriate basis element appears in Eq. (47) on the right-hand side. Hence, the amplitudes $b_n(t)$ of the irrotational modes should be varying in time.

IV. EXACT SOLUTIONS FOR THE MODAL AMPLITUDES

In the literature, wideband signals like the surges are often modeled by a double-exponential function; see [2], for example. We exhibit it here in the following form:

$$\mathfrak{s}(t) = (e^{-\gamma_1 t} - e^{-\gamma_2 t})/(e^{-\gamma_1 T} - e^{-\gamma_2 T}), \tag{48}$$

where $\gamma_2 > \gamma_1 > 0$ and T are three free parameters. As is easy to see, $\mathfrak{s}(T)=1$. The parameter T can be coupled with γ_1 and γ_2 by the condition of maximum: $\frac{d}{dt}\mathfrak{s}(t)|_{t=T}=0$. It yields $T=\eta/\gamma_1=\eta\rho/\gamma_2$ where

$$\rho = \gamma_2/\gamma_1 > 1, \quad \eta = (\ln \rho)/(\rho - 1) < 1. \tag{49}$$

This value of the parameter T can be used in formula (3) and in Eqs. (44)–(46) as the "intrinsic time."

In the numerical examples that follow, it is convenient to operate with a scaled (dimensionless) time, which we introduce for the surges as $\xi = t/T$. Then

$$\mathfrak{s}(\xi) = (e^{-\eta\xi} - e^{-\rho\eta\xi})e^{\eta}\rho/(\rho - 1). \tag{50}$$

This form of the function (with the variable ξ instead of t) has only one free parameter ρ. The dependence of the function $\mathfrak{s}(\xi)$ on the parameter ρ is shown in Fig. 1.

Consider as well the case when the function $\mathfrak{s}(t)$ in Eqs. (3) and (44)–(46) is habitually sinusoidal (for comparison). Denote then $\mathfrak{s}(t)$ as $\mathfrak{s}_1(t)$ and specify it as follows:

$$\mathfrak{s}_1(t) = \sin(\Omega t), \tag{51}$$

where Ω is a parameter. Then the period of oscillations, $T_\Omega = 2\pi/\Omega$, may be taken as the intrinsic time.

A. Exact solutions for the irrotational modes

An exact solution to problem (46) is the convolution integral

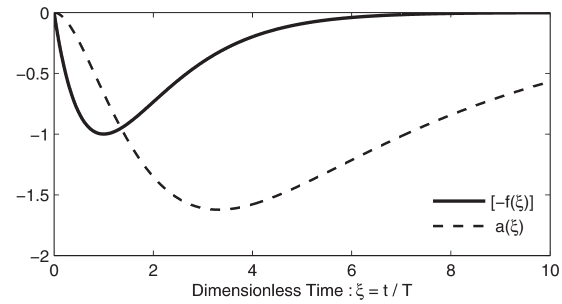

FIG. 2. Dependence on time of the surge $[-s(\xi)$ (solid line), $\rho=1.001]$ and normalized amplitude a_n.

$$a_n(\xi) = -F_n^0 H(\xi)\int_0^{\xi} e^{2\eta\rho_0(x-\xi)}\mathfrak{s}(x)dx, \tag{52}$$

where $\rho_0 = \gamma/\gamma_1$ and $\gamma = \sigma/(2\varepsilon_0)$ is the lossy parameter.

First consider excitation of the irrotational modes by the time-harmonic signal (51). To this aim, substitute $\mathfrak{s}_1(x) = \mathfrak{s}_1(2\pi\xi)|_{\xi=x}$ into the integrand in formula (52) instead of $\mathfrak{s}(x)$. In this case, time ξ is defined as t/T_Ω. The result obtained by formula (52) and exhibited in real time for convenience of analysis is

$$a_n(t) = \frac{F_n^0 H(t)}{2\pi\sqrt{1+(2\gamma/\Omega)^2}}\left[\cos(\Omega t + \theta) - \frac{\exp(-2\gamma t)}{\sqrt{1+(2\gamma/\Omega)^2}}\right], \tag{53}$$

where $\theta = \cos^{-1}[1/\sqrt{1+(2\gamma/\Omega)^2}]$, $n=1,2,\ldots$.

Observation of formula (53) suggests two conclusions. (a) The amplitudes $a_n(t)$ do not depend on the geometrical parameters of the cavity. Hence, the irrotational modes can be excited by signal $\mathfrak{s}_1(t)$ in any cavity. (b) The irrotational modes can oscillate with arbitrary frequency Ω of the applied signal without resonances.

The modal amplitudes $a_n(\xi)$, $n=1,2,\ldots$, excited by the surge signal (50) were calculated by the same formula (52). In Fig. 2, the results (normalized by the constant F_n^0) are exhibited.

B. Exact solutions for the solenoidal modes

Normalize the modal amplitudes in the problems (44) and (45) by the constants F_n' and F_n'', respectively. Inasmuch as the problems (44) and (45) are the same, mathematically, consider them in parallel. So the subscripts n and the primes are superfluous now. By making use of the dimensionless time $\xi = t/T$, the problems (44) and (45) are presented as

$$\frac{d}{d\xi}e(\xi) + 2\alpha e(\xi) - \breve{\varpi}h(\xi) = -H(\xi)\mathfrak{s}(\xi), \quad e(0) = 0, \tag{54a}$$

$$\frac{d}{d\xi}h(\xi) + \breve{\varpi}e(\xi) = 0, \quad h(0) = 0, \tag{54b}$$

where $\alpha = \gamma T$, $\breve{\varpi} = \omega T$, and ω is the eigenvalue ω_n' or ω_n''.

Reorganize the problem (54) to a vectorial form. To this aim, introduce a vector $X(\xi)$ composed of the modal amplitudes $e(\xi)$ and $h(\xi)$ as follows:

$$X(\xi) = \mathrm{col}(e(\xi), h(\xi)). \quad (55)$$

Problem (54) becomes equivalent to

$$\frac{d}{d\xi}X + QX = -\mathfrak{F}H(\xi)\mathfrak{s}(\xi), \quad X(0) = \mathrm{col}(0,0), \quad (56)$$

where $\mathfrak{F}=\mathrm{col}(1,0)$, and the coefficient Q is a 2×2 matrix:

$$Q = \begin{pmatrix} 2\alpha & -\breve{\varpi} \\ \breve{\varpi} & 0 \end{pmatrix}. \quad (57)$$

Formally, this problem was solved previously; see [7]. The final result of that solution [adapted here to the problem (56)] is the following vectorial convolution integral:

$$X(\xi) = -H(\xi)\int_0^{\xi}[e^{(x-\xi)Q} * \mathfrak{F}]\mathfrak{s}(x)dx, \quad (58)$$

where $e^{(x-\xi)Q}$ is a matrix function of the variable $(x-\xi)$.

The matrix exponential functions can be calculated by various methods [14]. Apply here an analytical method, which is based on Lagrange interpolation. Calculation begins with solving the characteristic equation for the matrix Q: $\det(\lambda U - Q)=0$ where U is the 2×2 identity matrix. It yields a pair of eigenvalues for the matrix Q as

$$\lambda_{1,2} = \alpha \pm \sqrt{\alpha^2 - \breve{\varpi}^2}. \quad (59)$$

It is easy to see that there are three different situations: *case 1*, $\breve{\varpi}^2 > \alpha^2$; *case 2*, $\breve{\varpi}^2 = \alpha^2$; and *case 3*, $\breve{\varpi}^2 < \alpha^2$. Consider these cases sequentially.

In cases 1 and 3 the eigenvalues are distinct, $\lambda_1 \neq \lambda_2$, but in case 2 they are degenerated, $\lambda_1 = \lambda_2$. The procedure of Lagrange interpolation is slightly different for these two situations. The result of interpolation valid for cases 1 and 3 (distinct eigenvalues) is

$$e^{(x-\xi)Q} = \frac{\lambda_2 U - Q}{\lambda_2 - \lambda_1}e^{(x-\xi)\lambda_1} + \frac{\lambda_1 U - Q}{\lambda_1 - \lambda_2}e^{(x-\xi)\lambda_2}. \quad (60)$$

Case 1: $\breve{\varpi}^2 > \alpha^2$. This situation is typical for microwave cavities. Denote $\sqrt{\alpha^2 - \breve{\varpi}^2} = i\varpi$ where i is an imaginary unit. Calculations by formulas (58) and (60) yield an exact solution as the convolution integrals: namely,

$$e(\xi) = -\frac{H(\xi)}{\cos\varphi}\int_0^{\xi} e^{\alpha(x-\xi)}\cos[(\xi - x)\varpi + \varphi]\mathfrak{s}(x)dx,$$

$$h(\xi) = -\frac{H(\xi)}{\cos\varphi}\int_0^{\xi} e^{\alpha(x-\xi)}\sin[(x - \xi)\varpi]\mathfrak{s}(x)dx, \quad (61)$$

where $\varpi = \sqrt{\breve{\varpi}^2 - \alpha^2} > 0$ and $\varphi = \cos^{-1}(\varpi/\breve{\varpi})$.

First consider excitation of the solenoidal modes by the surge (50). The results are shown in Fig. 3.

Now consider excitation of the solenoidal modes by the sinusoid (51). An exact solution (see Fig. 4) for the modal amplitudes is obtained explicitly in real time as

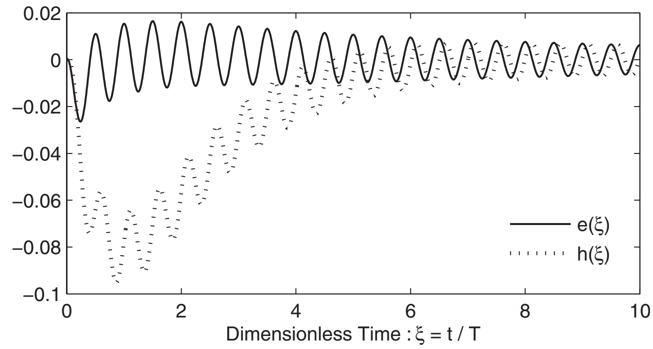

FIG. 3. Case 1: excitation by the surge, $\rho=1.001$. Amplitudes $e(\xi)$ and $h(\xi)$ as functions of time.

$$e(t) = -H(t)\frac{\omega}{2\omega_\gamma}\left[\frac{S_1(t)}{TR_-} + \frac{S_2(t)}{TR_+}\right],$$

$$h(t) = -H(t)\frac{\omega}{2\omega_\gamma}\left[\frac{C_1(t)}{TR_-} + \frac{C_2(t)}{TR_+}\right]. \quad (62)$$

A set of notation introduced in formulas (62) is

$$R_- = \sqrt{\gamma^2 + (\omega_\gamma - \Omega)^2}, \quad R_+ = \sqrt{\gamma^2 + (\omega_\gamma + \Omega)^2},$$

$$S_1(t) = \sin(\Omega t + \varphi + \vartheta) - e^{-\gamma t}\sin(\omega_\gamma t + \varphi + \vartheta),$$

$$S_2(t) = \sin(\Omega t + \varphi - \beta) + e^{-\gamma t}\sin(\omega_\gamma t - \varphi + \beta),$$

$$C_1(t) = \cos(\Omega t + \vartheta) - e^{-\gamma t}\cos(\omega_\gamma t + \vartheta),$$

$$C_2(t) = \cos(\Omega t - \beta) - e^{-\gamma t}\cos(\omega_\gamma t + \beta),$$

where $\omega_\gamma = \sqrt{\omega^2 - \gamma^2}$, $\gamma = \sigma/(2\varepsilon_0)$ is the lossy parameter, $\varphi = \sin^{-1}\frac{\gamma}{\omega}$, $\vartheta = \sin^{-1}\frac{\omega_\gamma - \Omega}{R_-}$, and $\beta = \sin^{-1}\frac{\omega_\gamma + \Omega}{R_+}$.

If the frequency Ω (a free parameter) is close to ω_γ, the terms $\frac{S_1(t)}{TR_-}$ and $\frac{C_1(t)}{TR_-}$ prevail over the terms $\frac{S_2(t)}{TR_+}$ and $\frac{C_2(t)}{TR_+}$ in solution (62). In our prior publications, the latter two terms were omitted.

A resonance occurs provided that the frequency Ω of the time-harmonic signal coincides with ω_γ, which yields

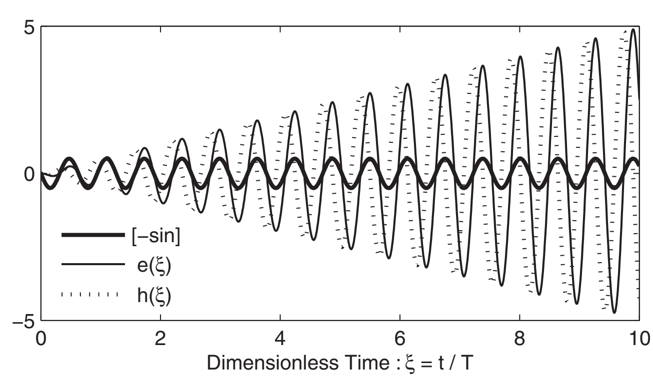

FIG. 4. Case 1: excitation by sinusoid. Amplitudes $e(\xi)$ and $h(\xi)$ provided that the condition of resonance holds.

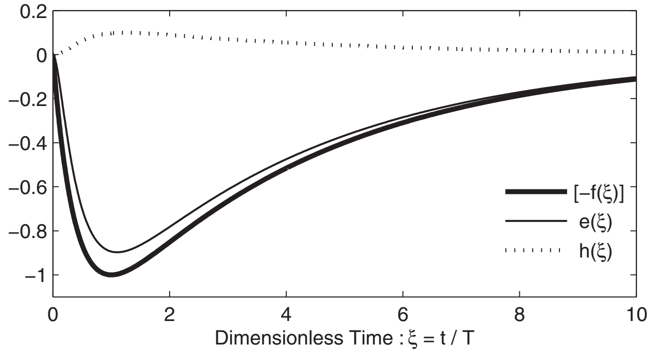

FIG. 5. Case 2: excitation by the surge, $\rho=1.001$. Amplitudes $e(\xi)$ and $h(\xi)$ as functions of time.

$$\Omega = \omega_\gamma = \sqrt{\omega^2 - \gamma^2}. \tag{63}$$

Recall that ω is an eigenvalue (ω_n' or ω_n'') of the operator $\mathfrak{R}$ and γ is the lossy parameter. If $\gamma=0$, then $\Omega=\omega$: i.e., the eigenvalue ω specifies a resonance frequency of the loss-free hollow cavity. If $\gamma\neq 0$, the quantity ω_γ is the resonant frequency of the cavity loaded by a lossy medium.

Under the condition of resonance, Eq. (63), the amplitudes of the solenoidal modes are almost equal to

$$e(t) \approxeq - H(t)\frac{\omega}{2\omega_\gamma}\frac{1-e^{-\gamma t}}{\gamma T}\sin(\omega_\gamma t + \varphi),$$

$$h(t) \approxeq - H(t)\frac{\omega}{2\omega_\gamma}\frac{1-e^{-\gamma t}}{\gamma T}\cos(\omega_\gamma t), \tag{64}$$

provided that $\gamma \ll 2\omega_\gamma$. If $\gamma \to 0$ and Ω coincides with the eigenvalue ω, formulas (64) become very simple:

$$\left.\begin{matrix} e(t) \\ h(t) \end{matrix}\right\} \approxeq - H(t)\left(\frac{t}{2T}\right) \times \left\{\begin{matrix} \sin(\omega t), \\ \cos(\omega t). \end{matrix}\right. \tag{65}$$

If the condition (63) does not hold, some beatings occur between oscillation of the signal with frequency Ω and decaying cavity oscillation with frequency ω_γ; see above the formulas for $S_{1,2}(t)$ and $C_{1,2}(t)$.

Case 2: $\breve{\varpi}^2=\alpha^2$ and, hence, $\lambda_1=\lambda_2=\alpha$. The procedure of Lagrange interpolation should be modified appropriately. Finally it yields the convolution integrals

$$e(\xi) = - H(\xi)\int_0^\xi e^{\alpha(x-\xi)}[1 + (x-\xi)\alpha]\mathfrak{s}(x)dx,$$

$$h(\xi) = - H(\xi)\int_0^\xi e^{\alpha(x-\xi)}(x-\xi)\alpha\mathfrak{s}(x)dx, \tag{66}$$

as the exact solution, where $\alpha=\gamma T=\omega T$. A numerical example for this case is presented in Fig. 5.

Case 3: $\breve{\varpi}^2<\alpha^2$. The eigenvalues of the matrix Q are distinct and real valued as $\lambda_1=\alpha+\delta$ and $\lambda_2=\alpha-\delta$ where $\delta=\sqrt{\alpha^2-\breve{\varpi}^2}>0$. It yields an exact solution in the form of the convolution integrals as

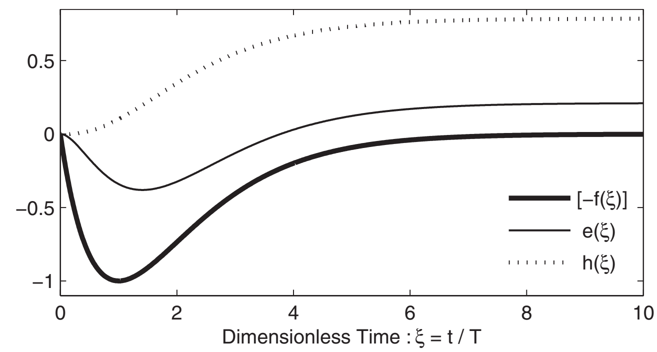

FIG. 6. Case 3: excitation by the surge, $\rho=1.001$. Amplitudes $e(\xi)$ and $h(\xi)$ as functions of time.

$$e(\xi) = -\frac{H(\xi)}{\sinh\theta}\int_0^\xi e^{\alpha(x-\xi)}\sinh[\delta(x-\xi)+\theta]\mathfrak{s}(x)dx,$$

$$h(\xi) = -\frac{H(\xi)}{\sinh\theta}\int_0^\xi e^{\alpha(x-\xi)}\sinh[\delta(x-\xi)]\mathfrak{s}(x)dx, \tag{67}$$

where $\theta=\sinh^{-1}(\sqrt{\alpha^2-\breve{\varpi}^2}/\breve{\varpi})$, $\alpha=\gamma T$, and $\breve{\varpi}=\omega T<\alpha$. A numerical example is presented in Fig. 6.

V. DISCUSSION

(i) The formulas for the electromagnetic fields, which can be excited in the cavity by a given signal, are derived as modal expansions with time-dependent modal amplitudes. Exact solutions are obtained explicitly for the amplitudes. They have the form of convolution integrals with time as the variable of integration. The integrands consist of elementary functions and the signal function, which stands as a parametric factor. The signal may be an arbitrary integrable function. Furthermore, one can substitute into the integrand an even approximation of a signal detected experimentally, generally speaking.

Two examples of the signal functions are considered: (a) the surge modeled by the double-exponential function of time and (b) the sinusoid having a beginning in time.

(ii) The verification of the theoretical method, which was used in the present study, was performed earlier by making use of the finite-difference time-domain method. One can find a comparison of the graphical results in [7] and [8]. The theoretical results and the numerical data almost coincide graphically.

(iii) The present time-domain study is limited by the case when the cavity is hollow (no any medium within it). The method can be extended to the cases, when the cavity is filled with a dispersive medium. It can be performed for Debye and Lorentz media. The system of Maxwell's equations with ∂_t should be solved simultaneously with appropriate motion equations. The latter play the role of dynamic constitutive relations. Some aspects of the applications of such dynamic constitutive relations in the time-domain electromagnetics were discussed in Ref. [8].

[1] A. Taflove and S. Hagness, *Computational Electrodynamics: The Finite-Difference Time-Domain Method* (Artech House, Boston, 2005).
[2] S. L. Dvorak and D. G. Dudley, IEEE Trans. Electromagn. Compat. **37**, 192 (1995).
[3] O. A. Tretyakov, in *Analytical and Numerical Methods in Electromagnetic Wave Theory*, edited by M. Hashimoto, M. Idemen, and O. A. Tretyakov (Science House, Tokyo, 1993), Chap. 3.
[4] O. A. Tretyakov, in *Evolutionary Equations for the Theory of Waveguides*, in Proceedings of the IEEE AP-S International Symposium Digest (Seattle, WA, 1994), pp. 2465–2471, http://ieeexplore.ieee.org/xpl/freeabs_all.jsp?isnumber=9161&arnumber=408101&count=235&index=90
[5] S. Aksoy and O. A. Tretyakov, J. Electromagn. Waves Appl. **16**, 1535 (2002).
[6] S. Aksoy and O. A. Tretyakov, J. Electromagn. Waves Appl. **17**, 263 (2004).
[7] S. Aksoy and O. A. Tretyakov, IEEE Trans. Antennas Propag. **52**, 263 (2004).
[8] S. Aksoy, M. Antyufeyeva, E. Basaran, A. A. Ergin, and O. A. Tretyakov, IEEE Trans. Microwave Theory Tech. **53**, 2465 (2005).
[9] W. Geyi, PIER **59**, 267 (2006).
[10] J. M. Cramer, R. A. Scholtz, and M. Z. Win, *On the Analysis of UWB Communication Channels*, in Proceedings of the IEEE Military Communications Conference (IEEE, New York, 1999), Vol. 2, pp. 1191–1195.
[11] C. L. Bennett and G. F. Ross, Proc. IEEE **66**, 299 (1978).
[12] *Ultra-Wideband Short-Pulse Electromagnetics*, edited by C. E. Baum, L. Carin, and A. P. Stone (Plenum Press, New York, 1997), Vol. 3.
[13] H. Weyl, Duke Math. J. **7**, 411 (1940).
[14] http://math.fullerton.edu/mathews/n2003/matrixexponential/MatrixExponentialBib/Links/MatrixExponentialBib_lnk_2.html